Numerical Methods
(MTH4002)
Lecture 08: Solution of Linear Systems

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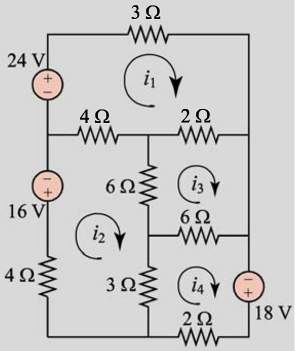
Introduction

- Systems of linear equations that have to be solved simultaneously arise in problems that include several (possibly many) variables that are dependent on each other.
- A system of two (or three) equations with two (or three) unknowns can be solved manually by substitution or other mathematical methods (e.g., Cramer's rule).
- Solving a system in this way is practically impossible as the number of equations (and unknowns) increases beyond three.

A Practical Example



- Using Kirchhoff's law, the currents i_1, i_2, i_3, and i_4 can be determined by solving the following system of four equations:

$$9i_1 - 4i_2 - 2i_3 = 24$$
$$-4i_1 + 17i_2 - 6i_3 - 3i_4 = -16$$
$$-2i_1 - 6i_2 + 14i_3 - 6i_4 = 0$$
$$-3i_2 - 6i_3 + 11i_4 = 18$$

Topics to be covered

- Vector, matrices and their properties
- Linear system of equations
- Upper triangular linear system
- Gaussian Elimination & Pivoting
- Triangular factorization
- Iterative methods for linear systems

Preliminaries

- Vector/Matrices and their properties
  - A vector has magnitude and direction. Vectors are useful in representing practical quantities.
  - In a generalized form, a vector x can be represented in n-dimensional space as

$$\mathrm{x} = (x_1, x_2, ..., x_n),$$

    where the numbers $x_1, x_2, \ldots, x_n$ are called the components or coordinates of vector x.
  - When a vector is used to denote a point or position in space, it is called a position vector.
  - When it is used to denote a movement between two points in space, it is called a displacement vector.

Preliminaries

- Let another vector be $\mathrm{y} = (y_1, y_2, \ldots, y_n)$. The two vectors y and x are said to be equal if and only if each corresponding coordinate is the same; that is,

$$\mathrm{x} = \mathrm{y} \quad \Leftrightarrow \quad x_j = y_j, \qquad \text{for} \quad j = 1, 2, \ldots, n. \tag{1}$$

- The sum of the vectors x and y is computed component by component.

$$\mathrm{x} + \mathrm{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \tag{2}$$

- The negative of the vector x is obtained by replacing each coordinate with its negative.

$$-\mathrm{x} = (-x_1, -x_2, \ldots, -x_n) \tag{3}$$

- The difference $\mathrm{x} - \mathrm{y}$ is formed by taking the difference in each coordinate:

$$\mathrm{y} - \mathrm{x} = (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n) \tag{4}$$

Preliminaries

- Vectors in n-dimensional space obey the algebraic property

$$\mathrm{y} - \mathrm{x} = \mathrm{y} + (-\mathrm{x}). \tag{5}$$

- If c is a real number (scalar), we define scalar multiplication $c\mathrm{x}$ as follows:

$$c\mathrm{x} = (cx_1, cx_2, \ldots, cx_n). \tag{6}$$

- If c and d are scalars, then the weighted sum $c\mathrm{x} + d\mathrm{y}$ is called a linear combination of x and y.

$$c\mathrm{x} + d\mathrm{x} = (cx_1 + dy_1, cx_2 + dy_2, \ldots, cx_n + dy_n) \tag{7}$$

Preliminaries

- The dot product of the two vectors x and y is a scalar quantity (real number) defined by the equation

$$\mathrm{x} \cdot \mathrm{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \tag{8}$$

- The norm (or length) of the vector x is defined by

$$|\mathrm{x}| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} \tag{9}$$

  Above equation is referred to as the Euclidean norm (or length) of the vector x.

- It is worth noting that

$$|\mathrm{x}|^2 = (x_1^2 + x_2^2 + \ldots + x_n^2) = \mathrm{x} \cdot \mathrm{x} \tag{10}$$

Preliminaries

- The distance travelled by a particle moving from points x to point y in n dimensional space is given by

$$|\mathrm{x}-\mathrm{y}| = ((y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2)^{1/2} \quad (11)$$

- Vector Algebra: Suppose that x, y, and rmz are n-dimensional vectors and a and b are scalars (real numbers). The following properties of vector addition and scalar multiplication hold:

$$\mathrm{y}+\mathrm{x} = \mathrm{x}+\mathrm{y} \quad \text{commutative property} \quad (12)$$

$$0+\mathrm{x} = \mathrm{x}+0 \quad \text{additive property} \quad (13)$$

$$\mathrm{x}-\mathrm{x} = \mathrm{x}+(-\mathrm{x}) \quad \text{additive inverse} \quad (14)$$

$$(\mathrm{x}+\mathrm{y})+\mathrm{z} = \mathrm{x}+(\mathrm{y}+\mathrm{z}) \quad \text{associative property} \quad (15)$$

$$(a+b)\mathrm{x} = a\mathrm{x}+b\mathrm{x} \quad \text{distributive property of scalars} \quad (16)$$

$$a(\mathrm{x}+\mathrm{y}) = a\mathrm{x}+a\mathrm{y} \quad \text{distributive property for vectors} \quad (17)$$

$$a(b\mathrm{x}) = (ab)\mathrm{x} \quad \text{associative property for scalars} \quad (18)$$

Matrices

- There is a close relationship between matrices and vectors.
- The matrix may be thought of as being composed of row vectors, or, alternatively, column vectors.
- A vector is a special case of a matrix.
- A row vector is simply a matrix with one row and several columns, and a column vector is simply a matrix with several rows and one column.

Matrices

- A matrix is a rectangular array of numbers that is arranged systematically in rows and columns.
- A matrix having m rows and n columns is called an $m \times n$ (read "m by n") matrix.
- The capital letter A denotes a matrix, and the lowercase subscripted letter a_{ij} denotes one of the numbers forming the matrix.

$$A = [a_{ij}]_{m\times n} \qquad \text{for } 1 \le i \le m, 1 \le j \le n, \tag{19}$$

where a_{ij} is the number in location $(i, j$ (i.e., stored in the i^{th} row and j^{th} column of the matrix). We refer to a_{ij} as the element in location (i, j).

Matrices

- In expanded form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{ml} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A. \tag{20}$$

Matrices

- Matrix Addition and Scalar multiplication: Suppose that A, B, and C are $m \times n$ matrices and p and q are scalars. The following properties of matrix addition and scalar multiplication hold

$$B + A = A + B \qquad \text{commutative property} \qquad (21)$$

$$0 + A = A + 0 \qquad \text{additive identity} \qquad (22)$$

$$A - A = A + (-A) = 0 \qquad \text{additive inverse} \qquad (23)$$

$$(A + B) + C = A + (B + C) \qquad \text{associative property} \qquad (24)$$

$$(p + q)A = pA + qA \qquad \text{distributive property for scalars} \qquad (25)$$

$$p(A + B) = pA + pB \qquad \text{distributive property for matrices} \qquad (26)$$

$$p(qA) = (pq)A \qquad \text{associative property for scalars} \qquad (27)$$

Special Matrices

- Square matrix
- Diagonal matrix
- Upper triangular matrix
- Lower triangular matrix
- Identity matrix
- Zero matrix
- Symmetric matrix

Cramer's Rule

- A set of n simultaneous linear equations with n unknowns $x_1, x_2, \ldots, x_n$ is given by:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \quad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

- The system can be written compactly by using matrices:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{28}$$

Cramer's Rule

- The system or set of equation can also be written as

$$A \cdot \mathrm{x} = \mathrm{b} \qquad \text{or} \qquad [A][\mathrm{x}] = [\mathrm{b}]$$

where A is the matrix of coefficients, x is the vector of n unknowns, and b is the vector containing the right-hand sides of each equation.

- Cramer's rule states that the solution to set of linear equations, if it exists, is given by:

$$x_j = \frac{\det(A'_j)}{\det(A)} \quad \text{for} \;\; j = 1, 2, \ldots, n$$

where A'_j is the matrix formed by replacing the jth column of the matrix A with the column vector b.

Criteria to exist the solution

- Solutions can exist only if $\det(A) \neq 0$.
- The only way that $\det(A)$ can be zero is either
  - if two or more columns or rows of A are identical or
  - one or more columns (or rows) of A are linearly dependent on other columns (or rows).

Example

Concrete (used for sidewalks, etc.) is a mixture of portland cement, sand, and gravel. A distributor has three batches available for contractors. Batch 1 contains cement, sand, and gravel mixed in the proportions $1/8$, $3/8$, $4/8$; batch 2 has the proportions $2/10$, $5/10$, $3/10$; and batch 3 has the proportions $2/5$, $3/5$, $0/5$. For constructing a sidewalk of 10 cubic yards how much cubic yards of each batch to be mixed such that the mixture contains 2.3, 4.8, and 2.9 cubic yards of portland cement, sand, and gravel, respectively?

Overview of Numerical Methods for Solving SLAE

- Two types of numerical methods are used for solving systems of linear algebraic equations:
  - Direct method
  - Iterative method
- In direct methods, the solution is calculated by performing arithmetic operations with the equations.
- In iterative methods, an initial approximate solution is assumed and then used in an iterative process for obtaining successively more accurate solutions.

Direct methods

- In direct methods, the solution is calculated by performing arithmetic operations with the equations.
- The system of equations that is initially given in the general form is manipulated to an equivalent system of equations that can be easily solved.
- Three systems of equations (equivalent) that can be easily solved are
  - Upper triangular,
  - Lower triangular, and
  - Diagonal forms.
- Three direct methods for solving systems of equations
  1. Gauss elimination,
  2. Gauss-Jordan, and
  3. LU decomposition

Upper triangular

- The upper triangular form can be written in a matrix form for a system of four equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{44}x_4 &= b_4 \end{aligned} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- The system in this form has all zero coefficients below the diagonal.
- Can be solved by a procedure called back substitution.
- It starts with the last equation, which is solved for x_4. The value of x_4 is then substituted in the next-to-the-last equation, which is solved for x_3. The process continues in the same manner all the way up to the first equation.

Upper triangular

- In the case of four equations, the solution is given by:

$$x_4 = \frac{b_4}{a_{44}}, \quad x_3 = \frac{b_3 - a_{34}x_4}{a_{33}}, \quad x_2 = \frac{b_2 - (a_{23}x_3 + a_{24}x_4)}{a_{22}}, \quad \text{and}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)}{a_{11}}$$

- For a system of n equations in upper triangular form, a general formula for the solution using back substitution is

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{j=n} a_{ij}x_j}{a_{ii}} \quad i = n-1, n-2, \ldots, 1$$

- The upper triangular form and back substitution are used in the Gauss elimination method.

Lower triangular

- The lower triangular form can be written in a matrix form for a system of four equations as

$$\begin{aligned} a_{11}x_1 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \quad\quad\quad\quad\quad\quad &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \quad\quad &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- The system in this form has zero coefficients above the diagonal.
- Can be solved by a procedure called forward substitution.
- It starts with the first equation, which is solved for x_1. The value of x_1 is then substituted in the second equation, which is solved for x_2. The process continues in the same manner all the way down to the last equation.

Lower triangular

- In the case of four equations, the solution is given by:
$x_1 = \dfrac{b_1}{a_{11}}, \quad x_2 = \dfrac{b_2 - a_{21}x_1}{a_{22}}, \quad x_3 = \dfrac{b_3 - (a_{31}x_1 + a_{32}x_2)}{a_{33}},$ and
$x_4 = \dfrac{b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3)}{a_{44}}$
- For a system of n equations in lower triangular form, a general formula for the solution using forward substitution is:

$$x_1 = \frac{b_1}{a_{11}}$$
$$x_i = \frac{b_i - \sum_{j=1}^{j=i-1} a_{ij}x_j}{a_{ii}} \qquad i = 2, 3, \ldots, n$$

Diagonal triangular

- The diagonal form of a system of linear equations and the matrix form for system of four equation is given below

$$\begin{aligned} a_{11}x_1 \quad\quad\quad\quad\quad\quad &= b_1 \\ a_{22}x_2 \quad\quad\quad\quad &= b_2 \\ a_{33}x_3 \quad\quad &= b_3 \\ a_{44}x_4 &= b_4 \end{aligned} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Example

Question: Use back substitution to solve the linear system

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

Example

Question: Show that there is no solution to the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

Example

Question: Show that there are infinitely many solutions to

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 - 0x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

Gauss Elimination Method

- The Gauss elimination method is a procedure for solving a system of linear equations.
- In this procedure, a system of equations that is given in a general form is manipulated to be in upper triangular form, which is then solved by using back substitution.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4
\end{aligned}
\qquad
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}
\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4 \end{bmatrix}
$$

Gauss Elimination Method

- The system of equations is manipulated into an equivalent system of equations that has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 &= b'_2 \\ a'_{33}x_3 + a'_{34}x_4 &= b'_3 \\ a'_{44}x_4 &= b'_4 \end{aligned}$$

- The matrix form of the equivalent system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a'_{33} & a'_{34} \\ 0 & 0 & 0 & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Gauss elimination procedureL: STEP 1 (forward elimination)

- Converting the system of equations to the upper triangular form is done in following steps.
  Step 1: In the first step, the first equation is unchanged, and the terms that include the variable x_1 in all the other equations are eliminated. This is done one equation at a time by using the first equation, which is called the pivot equation. The coefficient a_{11} is called the pivot coefficient, or the pivot element.

$$\begin{array}{rl} & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ - & m_{21}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{21}b_1 \\ \hline & 0 + \underbrace{(a_{22} - m_{21}a_{12})}_{a'_{22}}x_2 + \underbrace{(a_{23} - m_{21}a_{13})}_{a'_{23}}x_3 + \underbrace{(a_{24} - m_{21}a_{14})}_{a'_{24}}x_4 = \underbrace{b_2 - m_{21}b_1}_{b'_2} \end{array}$$

Gauss elimination procedure

- To eliminate the term $a_{21}x_1$ in the pivot equation, The first equation is multiplied by $m_{21} = a_{21}/a_{11}$, and then the equation is subtracted to second equation.
- It should be emphasized here that the pivot equation itself is not changed.
- The matrix form of the equations after this operation is shown as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- This process repeats in the same manner to eliminate the lower triangle elements to zero.

Gauss elimination procedure

- The first equation is multiplied by $m_{31} = a_{31}/a_{11}$

$$
\begin{array}{rl}
 & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\
- & m_{31}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{31}b_1 \\
\hline
 & 0 + \underbrace{(a_{32} - m_{31}a_{12})}_{a'_{32}}x_2 + \underbrace{(a_{33} - m_{31}a_{13})}_{a'_{33}}x_3 + \underbrace{(a_{34} - m_{31}a_{14})}_{a'_{34}}x_4 = \underbrace{b_3 - m_{31}b_1}_{b'_3}
\end{array}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b_4 \end{bmatrix}
$$

- The first equation is multiplied by $m_{41} = a_{41}/a_{11}$

$$
\begin{array}{rl}
 & a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \\
- & m_{41}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{41}b_1 \\
\hline
 & 0 + \underbrace{(a_{42} - m_{41}a_{12})}_{a'_{42}}x_2 + \underbrace{(a_{43} - m_{41}a_{13})}_{a'_{43}}x_3 + \underbrace{(a_{44} - m_{41}a_{14})}_{a'_{44}}x_4 = \underbrace{b_4 - m_{41}b_1}_{b'_4}
\end{array}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}
$$

Gauss elimination procedure

- Step 2: In this step, first two equation do not change.
- The terms that include the variable x_2 in rest of the equations are eliminated.
- The second equation is multiplied by $m_{32} = a'_{32}/a'_{22}$ and subtracted

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

$$\begin{array}{rl} & a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 = b'_3 \\ - & m_{32}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) = m_{32}b'_2 \\ \hline & 0 + \underbrace{(a'_{33} - m_{32}a'_{23})}_{a''_{33}}x_3 + \underbrace{(a'_{34} - m_{32}a'_{24})}_{a''_{34}}x_4 = \underbrace{b'_3 - m_{32}b'_2}_{b''_3} \end{array}$$

- The second equation is multiplied by $m_{42} = a'_{42}/a'_{22}$ and subtracted

$$\begin{array}{rl} & a'_{42}x_2 + a'_{43}x_3 + a'_{44}x_4 = b'_4 \\ - & m_{42}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) = m_{42}b'_2 \\ \hline & 0 + \underbrace{(a'_{43} - m_{42}a'_{23})}_{a''_{43}}x_3 + \underbrace{(a'_{44} - m_{42}a'_{24})}_{a''_{44}}x_4 = \underbrace{b'_4 - m_{42}b'_2}_{b''_4} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b''_4 \end{bmatrix}$$

Gauss elimination procedure

- Step 4: In this step, first three equation do not change.
- The terms that include the variable x_3 in rest of the equations are eliminated.
- The third equation is multiplied by $m_{43} = a'_{43}/a'_{33}$ and subtracted

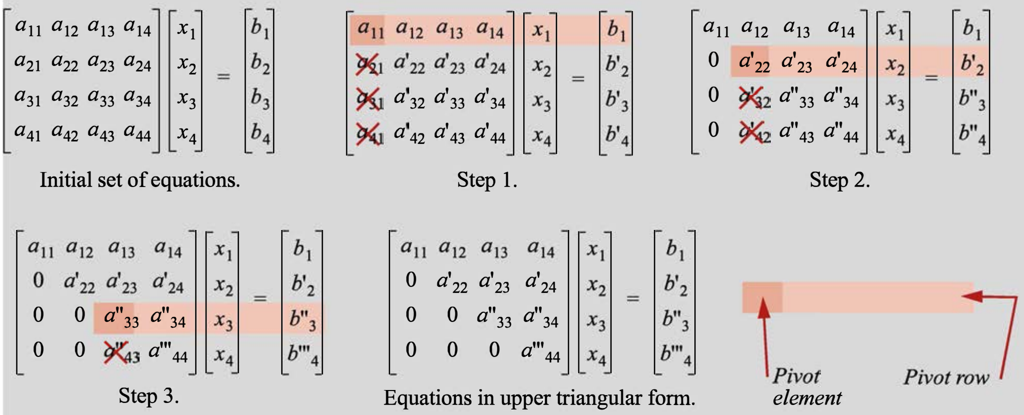
$$
\begin{array}{rrcl}
 & a''_{43}x_3 + a''_{44}x_4 & = & b''_4 \\
- & m_{43}(a''_{33}x_3 + a''_{34}x_4) & = & m_{43}b''_3 \\
\hline
 & \underbrace{(a''_{44} - m_{43}a''_{34})}_{a'''_{44}}x_4 & = & \underbrace{b''_4 - m_{43}b''_3}_{b'''_4}
\end{array}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}
$$

- The system of equations is now in an upper triangular form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\
0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 &= b'_2 \\
0 + \quad 0 \quad + a''_{33}x_3 + a''_{34}x_4 &= b''_3 \\
0 + \quad 0 \quad + \quad 0 \quad + a'''_{44}x_4 &= b'''_4
\end{aligned}
$$

Gauss elimination procedure



- Once transformed to upper triangular form, the equations can be easily solved by using back substitution.

Problem while applying the Gauss Elimination Method

- There are some potential difficulties when applying the Gauss elimination method
  - The pivot element is zero: Since the pivot row is divided by the pivot element, a problem will arise during the execution of the Gauss elimination procedure if the value of the pivot element is equal to zero. In a procedure called pivoting, the pivot row that has the zero pivot element is exchanged with another row that has a nonzero pivot element.
  - The pivot element is small relative to the other terms in the pivot row: Significant errors due to rounding can occur when the pivot element is small relative to other elements in the pivot row.

Gauss elimination with pivoting

- In the Gauss elimination procedure, the pivot equation is divided by the pivot coefficient. This, however, cannot be done if the pivot coefficient is zero.

$$\begin{aligned} 0x_1 + 2x_2 + 3x_3 &= 46 \\ 4x_1 - 3x_2 + 2x_3 &= 16 \\ 2x_1 + 4x_2 - 3x_3 &= 12 \end{aligned}$$

- The division by zero can be avoided if the order in which the equations are written is changed such that in the first equation the first coefficient is not zero.
- For example, in the system above, this can be done by exchanging the first two equations.

Gauss elimination with pivoting

- All the coefficients of the linear system $A\mathrm{x} = \mathrm{b}$ can be stored in an augmented matrix, denoted as $[A \mid \mathrm{b}]$, of dimension $n \times (n+1)$. The linear system is represented as follows:

$$[A \mid \mathrm{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \quad (29)$$

- The system $A\mathrm{x} = \mathrm{b}$, with augmented matrix, can be solved by performing row operations on the augmented matrix $[A \mid \mathrm{b}]$.

Gauss elimination with pivoting

- Elementary Row Operations: The following operations applied to the augmented matrix that yield an equivalent linear system.
  1. Interchanges: The order of two rows can be changed.
  2. Scaling: Multiplying a row by a nonzero constant.
  3. Replacement: The row can be replaced by the sum of that row and a nonzero multiple of any other row; that is: $row_r = row_r - m_{rp} \times row_p$
- Use these operations to obtain an equivalent upper-triangular system $U\mathrm{x} = \mathrm{y}$ from a linear system $A\mathrm{x} = \mathrm{b}$, where A is an $n \times n$ matrix.

Example

Question: Express the following system in augmented matrix form and find an equivalent upper-triangular system and the solution.

$$\begin{aligned} x_1 \quad +2x_2 \quad +x_3 + 4x_4 &= 13 \\ 2x_1 \quad +0x_2 \quad +4x_3 + 3x_4 &= 28 \\ 4x_1 \quad +2x_2 \quad +2x_3 + x_4 &= 20 \\ -3x_1 \quad +x_2 \quad +3x_3 + 2x_4 &= 6 \end{aligned} \tag{30}$$

Example

Question: Express the following system in augmented matrix form and find an equivalent upper-triangular system and the solution.

$$\begin{aligned} 0x_1 \quad +2x_2 \quad +4x_3 + 3x_4 \quad &= 28 \\ 2x_1 \quad +1x_2 \quad +x_3 + 4x_4 \quad &= 13 \\ 2x_1 \quad +4x_2 \quad +2x_3 + x_4 \quad &= 20 \\ 1x_1 \quad -3x_2 \quad +3x_3 + 2x_4 \quad &= 6 \end{aligned} \tag{31}$$

Triangular Factorization

- The Gauss elimination method consists of two parts.
  - The first part is the elimination procedure.
  - In the second part, the equivalent system is solved by using back substitution
- The elimination procedure requires many mathematical operations and significantly more computing time than the back substitution calculations.
- During the elimination procedure, the matrix of coefficients A and the vector b are both changed.
- This means that if there is a need to solve systems of equations that have the same left-hand-side terms (same coefficient matrix A) but different right-hand-side constants (different vectors b), the elimination procedure has to be carried out for each b again.
- Ideally, it would be better if the operations on the matrix of coefficients A were dissociated from those on the vector of constants b.

Triangular Factorization

- In this way, the elimination procedure with A is done only once and then is used for solving systems of equations with different vectors b.
- One option for solving various systems of equations

$$A\mathrm{x} = \mathrm{b}$$

that have the same coefficient matrices A but different constant vectors b is to first calculate the inverse of the matrix A . Once the inverse matrix A^{-1} is known, the solution can be calculated by:

$$\mathrm{x} = A^{-1}\mathrm{b}$$

- Calculating the inverse of a matrix, however, requires many mathematical operations, and is computationally inefficient.

Triangular Factorization

- A more efficient method of solution for this case is the LU decomposition method.
- The LU decomposition method is a method for solving a system of linear equations $A\mathrm{x} = \mathrm{b}$
- In this method, the matrix of coefficients A is decomposed (factored) into a product of two matrices L and U:

$$A = LU$$

where the matrix L is a lower triangular matrix and U is an upper triangular matrix.

Triangular Factorization

- The nonsingular matrix A has a triangular factorization if it can be expressed as the product of a lower-triangular matrix L and an upper-triangular matrix U:

$$A = LU \tag{32}$$

In matrix form, this is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & w_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \tag{33}$$

The condition that A is nonsingular implies that $u_{kk} = 0$ for all k. The notation for the entries in L is m_{ij}.

Triangular Factorization

- So, we have

$$LU\mathrm{x} = \mathrm{b}. \tag{34}$$

- We can define $\mathrm{y} = U\mathrm{x}$ and then solve the two systems:

$$\text{first solve} \quad L\mathrm{y} = \mathrm{b} \quad \text{for} \quad \mathrm{y} \tag{35}$$
$$\text{then solve} \quad U\mathrm{x} = \mathrm{y} \quad \text{for} \quad \mathrm{x} \tag{36}$$

- In equation form, we must first solve the lower-triangular system

$$\begin{aligned} y_1 &= b_1 \\ m_{21}y_1 + y_2 &= b_2 \\ m_{31}y_1 + m_{32}y_2 + y_3 &= b_3 \\ m_{41}y_1 + m_{42}y_2 + m_{43}y_3 + y_4 &= b_4 \end{aligned} \tag{37}$$

Triangular Factorization

- Compute y_1, y_2, y_3, and y_4 and use them in solving the upper-triangular system

$$\begin{aligned} a_{11}x_1 + u_{12}x_2 + u_{13}x_3 + u_{14}x_4 &= y_1 \\ u_{22}x_2 + u_{23}x_3 + u_{24}x_4 &= y_2 \\ u_{33}x_3 + u_{34}x_4 &= y_3 \\ u_{44_{x_4}} &= y_4 \end{aligned} \tag{38}$$

LU Decomposition Using the Gauss Elimination Procedure

Solve the following system of equation using the triangular factorization method.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 0.25 & 2 & 1 & 0 \\ -3 & 4 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & -3 & 6 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Example

Solve the following system of equation using the triangular factorization method.

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\ 2x_1 + 8x_2 + 6x_3 + 4x_4 &= 52 \\ 3x_1 + 10x_2 + 8x_3 + 8x_4 &= 79 \\ 4x_1 + 12x_2 + 10x_3 + 6x_4 &= 82 \end{aligned} \tag{39}$$

Given

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 8 & 6 & 4 \\ 3 & 10 & 8 & 8 \\ 4 & 12 & 10 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix} = LU$$

Example

Answer: Use the forward-substitution method to solve $LY = B$:

$$\begin{aligned} y_1 &= 21 \\ 2y_1 + y_2 &= 52 \\ 3y_1 + y_2 + y_3 &= 79 \\ 4y_1 + y_2 + 2y_3 + y_4 &= 82 \end{aligned}$$

Compute the values $y_1 = 21$, $y_2 = 52 - 2(21) = 10$, $y_3 = 79 - 3(21) - 10 = 6$, and $y_4 = 82 - 4(21) - 10 - 2(6) = -24$, or $Y = [21\ 10\ 6\ \ -24]'$. Next write the system $UX = Y$:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\ 4x_2 - 2x_3 + 2x_4 &= 10 \\ -2x_3 + 3x_4 &= 6 \\ -6x_4 &= -24 \end{aligned}$$

Now use back substitution and compute the solution $x_4 = -24/(-6) = 4$, $x_3 = (6 - 3(4))/(-2) = 3$, $x_2 = (10 - 2(4) + 2(3))/4 = 2$, and $x_1 = 21 - 4 - 4(3) - 2(2) = 1$, or $X = [1\ 2\ 3\ 4]'$.

Iterative Method

- In iterative methods, an initial approximate solution is assumed and then used in an iterative process for obtaining successively more accurate solutions.
- Two indirect (iterative) methods are
  - Jacobi, and
  - Gauss-Seidel

Jacobi iterative method

Question: Consider the system of equations

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15 \end{aligned} \tag{40}$$

Solve using Jacobi method.

These equations can be written in the form

$$x = \frac{7 + y - z}{4} \qquad y = \frac{21 + 4x + z}{8} \qquad z = \frac{15 + 2x - y}{5}$$

Jacobi iterative method

- This suggests the following Jacobi iterative process:

$$x_{k+1} = \frac{7 + y_k - z_k}{4} \qquad y_{k+1} = \frac{21 + 4x_k + z_k}{8} \qquad z_{k+1} = \frac{15 + 2x_k - y_k}{5}$$

- Let us start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then check that solution converge to the solution $(2, 4, 3)$.
- Substitute $x_0 = 1$, $y_0 = 2$, and $z_0 = 2$ into the each equation and obtain the new values

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75 \quad y_1 = \frac{21 + 4 + 2}{8} = 3.375 \quad z_1 = \frac{15 + 2 - 2}{5} = 3.00 \tag{41}$$

The new point $P_1 = (1.75, 3.375, 3.00)$ is closer to $(2, 4, 3)$ than P_0.

Jacobi iterative method

- Table shows the convergence

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

Jacobi iterative method

- Linear systems with as many as 100,000 variables often arise in the solution of partial differential equations.
- The coefficient matrices for these systems are sparse; that is, a large percentage of the entries of the coefficient matrix are zero.
- If there is a pattern to the nonzero entries (i.e., tridiagonal systems), then an iterative process provides an efficient method for solving these large systems.
- Sometimes the Jacobi method does not work. Let see through an example.

Jacobi iterative method

Question: Let the linear system defined in previous example be rearranged as follows:

$$\begin{aligned} -2x + y + 5z &= 15 \\ 4x - 8y + z &= -21 \\ 4x - y + z &= 7 \end{aligned}$$

These equations can be written in the form

$$x = \frac{-15 + y + 5z}{3} \qquad y = \frac{21 + 4x + z}{8} \qquad z = 7 - 4x + y$$

This suggests the following Jacobi iterative process:

$$x_{k+1} = \frac{-15 + y_k + 5z_k}{3} \qquad y_{k+1} = \frac{21 + 4x_k + z_k}{8} \qquad z_{k+1} = 7 - 4x_k + y_k$$

Jacobi iterative method

- If we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$ then solution will diverge away from the solution $(2, 4, 3)$.

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮

Criterion for convergence

- In view of examples solved using Jacobi iterative method, it is necessary to have some criterion to determine whether the Jacobi iteration will converge. Hence we make the following definition.

Definition

A matrix A of dimension $n \times n$ is said to be strictly diagonally dominant provided that

$$|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \qquad \text{for } k = 1, 2, \ldots, n \tag{42}$$

- This means that in each row of the matrix the magnitude of the element on the main diagonal must exceed the sum of the magnitudes of all other elements in the row.

Criterion for convergence

- The coefficient matrix of the linear system in Example solved using Jacobi is strictly diagonally dominant because

$$\begin{aligned} \text{In row } 1: &\quad |4| > |-1| + |1| \\ \text{In row } 2: &\quad |-8| > |4| + |1| \\ \text{In row } 3: &\quad |5| > |-2| + |1| \end{aligned}$$

- The coefficient matrix A of the linear system in the Example, which is not converged to the solution, is not strictly diagonally dominant because

$$\begin{aligned} \text{In row } 1: &\quad |-2| < |1| + |5| \\ \text{In row } 2: &\quad |-8| > |4| + |1| \\ \text{In row } 3: &\quad |1| < |4| + |-1| \end{aligned}$$

Gauss-Seidel Iteration

- Since x_{k+1} is expected to be a better approximation to x than x_k.
- It is reasonable that x_{k+1} could be used in place of x_k in the computation of y_{k+1}.
- Similarly, x_{k+1} and y_{k+1} might be used in the computation of z_{k+1}.
- Let us solve the previous example to understand the process of Gauss-Seidel iteration.
- Gauss-Seidel Iteration considers the following system of equations for previous example

$$x_{k+1} = \frac{7 + y_k - z_k}{4} \qquad y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8} \qquad z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}$$

Gauss-Seidel Iteration

- If we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then iteration using Gauss-Seidel will converge to the solution $(2, 4, 3)$.
- Substitute $y_0 = 2$ and $z_0 = 2$ into the first equation and obtain

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75 \tag{43}$$

- Then substitute $x_1 = 1.75$ and $z_0 = 2$

$$y_1 = \frac{21 + 4(1.75) + 2}{8} = 3.75 \tag{44}$$

- Finally, substitute $x_1 = 1.75$ and $y_1 = 3.75$ into the third equation to get

$$z_1 = \frac{15 + 2(1.75) - 3.75}{5} = 2.95 \tag{45}$$

Gauss-Seidel Iteration

- The new point $P_1 = (1.75, 3.75, 2.95)$ is closer to $(2, 4, 3)$ than P_0 and is better estimate than the value obtained using Jacobi iterative method.

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.75	2.95
2	1.95	3.96875	2.98625
3	1.995625	3.99609375	2.99903125
⋮	⋮	⋮	⋮
8	1.99999983	3.99999988	2.99999996
9	1.99999998	3.99999999	3.00000000
10	2.00000000	4.00000000	3.00000000

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Thank you!