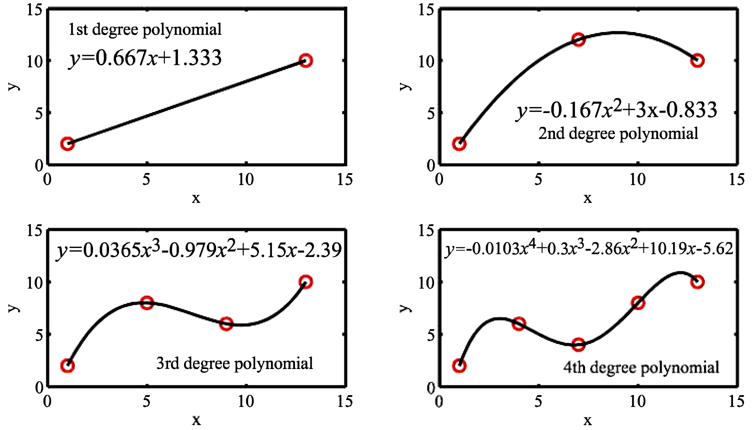
Numerical Methods
(MTH4002)
Lecture 04: Interpolation

Dr. Kundan Kumar
Associate Professor
Department of ECE

Faculty of Engineering (ITER)
S'O'A Deemed to be University, Bhubaneswar, India-751030

Outline

Interpolation



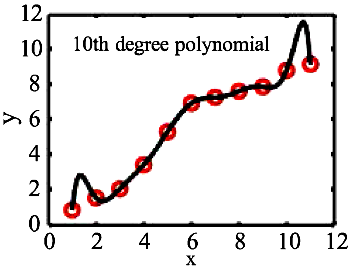
- Interpolation is a procedure for estimating a value between known values of data points.
- It is done by first determining a polynomial that gives the exact value at the data points, and then using the polynomial for calculating values between the points.

Interpolation vs. Curve Fitting

- Curve fitting
  - A procedure in which a mathematical formula (equation) is used to best fit a given set of data points.
  - The objective is to find a function that fits the data points overall.
  - This means that the function does not have to give the exact value at any single point, but fits the data well overall.
- Interpolation
  - A procedure for estimating a value between known values of data points.
  - Done by first determining a polynomial that gives the exact value at the data points, and then using the polynomial for calculating values between the points.
  - Interpolation passes through all the selected data points.
  - Interpolation works best when the data points have no error in them.

Interpolation

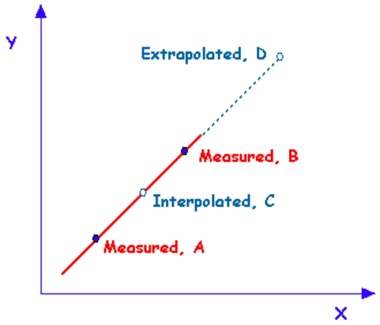
- Interpolation with a single polynomial gives good results for a small number of points.
- For a large number of points the order of the polynomial is high, and although the polynomial passes through all the points, it might deviate significantly between the points.



- Consequently, interpolation with a single polynomial might not be appropriate for a large number of points.
- For a large number of points, better interpolation can be done by using piecewise (spline) interpolation in which different lower-order polynomials are used for interpolation between different points of the same set of data points.

Interpolation and Extrapolation

- Sometimes the data points are used for estimating the expected values between the known points, a procedure called interpolation,
- For predicting how the data might extend beyond the range over which it was measured, a procedure called extrapolation.



Interpolation

- For a given set of n points, only one (unique) polynomial of order m $(m = n - 1)$ passes exactly through all of the points.
- The polynomial, however, can be written in different mathematical forms.
- Three forms of polynomials are
  - Standard,
  - Lagrange, and
  - Newton's
- Standard form of an mth-order polynomial is:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$

- The coefficients in this form are determined by solving a system of $n = (m+1)$ linear equations. (already discussed in curve fitting)
- Even though the high-order polynomial gives the exact values at all the data points, it cannot be used reliably for interpolation or extrapolation.

Lagrange Interpolating Polynomials

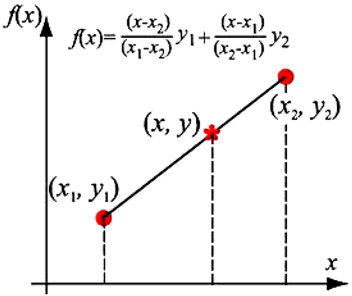
Lagrange Interpolating Polynomials

- Lagrange interpolating polynomials are a particular form of polynomials that can be written to fit a given set of data points by using the values at the points.
- The polynomials can be written right away and do not require any preliminary calculations for determining coefficients.
- Lagrange polynomials
  - First-order Lagrange polynomial
  - Second-order Lagrange polynomial
  - General form of an $(n-1)$ order Lagrange polynomial

First order Lagrange polynomial

- For two points, (x_1, y_1), and (x_2, y_2), the first-order Lagrange polynomial that passes through the points has the form:

$$f(x) = y = a_1(x - x_2) + a_2(x - x_1)$$



- Substitute the two point in the above equation

$$y_1 = a_1(x_1 - x_2) + a_2(x_1 - x_1) \quad \text{or } a_1 = \frac{y_1}{(x_1 - x_2)}$$
$$y_2 = a_1(x_2 - x_2) + a_2(x_2 - x_1) \quad \text{or } a_2 = \frac{y_2}{(x_2 - x_1)}$$

- Substitute a_1 and a_2 back

$$f(x) = \frac{(x - x_2)}{(x_1 - x_2)} y_1 + \frac{(x - x_1)}{(x_2 - x_1)} y_2$$

$$\Rightarrow f(x) = \frac{(y_2 - y_1)}{(x_2 - x_1)} x + \frac{x_2 y_1 - x_1 y_2}{(x_2 - x_1)}$$

Second-order Lagrange polynomial

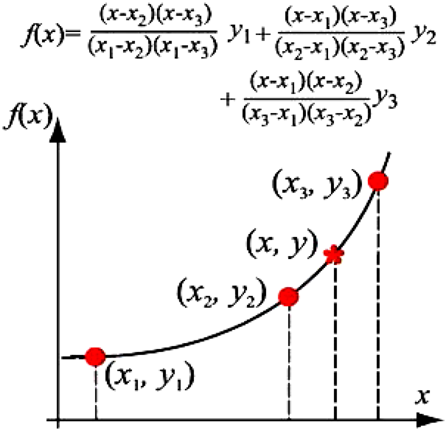
- For three points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), the second-order Lagrange polynomial that passes through the points has the form:

$$f(x) = y = a_1 (x - x_2)(x - x_3) + a_2 (x - x_1)(x - x_3) + a_3 (x - x_1)(x - x_2)$$

- Once the coefficients are determined such that the polynomial passes through the three points, the polynomial (quadratic form) is:

$$f(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} y_3$$

Second-order Lagrange polynomial



- When the coordinate x_1, x_2, or x_3 of one of the three given points is substituted in previous equation, the value of the polynomial is equal to y_1, y_2, or y_3, respectively.
- This is because the coefficient in front of the corresponding y_i is equal to 1 and the coefficient of the other two terms is equal to zero.
- The equation can also be rewritten in the standard form $f(x) = b_2x^2 + b_1x + b_0$. (You can easily derive this standard form)

General form of Lagrange polynomial

- The general formula of an $(n-1)$ order Lagrange polynomial that passes through n points (x_1, y_1), (x_2, y_2), ... , (x_n, y_n) is:

$$f(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \ldots + \frac{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}y_i + \ldots + \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n \tag{1}$$

$$f(x) = \sum_{i=1}^{n} y_i L_i(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \tag{2}$$

where $L_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$ are called the Lagrange functions.

Conclusive remarks

- The spacing between the data points does not have to be equal.
- For a given set of points, the whole expression of the interpolation polynomial has to be calculated for every value of x. In other words, the interpolation calculations for each value of x are independent of others. used for calculating different values of x.
- If an interpolated value is calculated for a given set of data points, and then the data set is enlarged to include additional points, all the terms of the Lagrange polynomial have to be calculated again.
- As discussed in next topic, this is different from Newton's polynomials where only the new terms have to be calculated if more data points are added.

Example

Example: The set of the following five data points is given:

x	1	2	4	5	7
y	52	5	-5	-40	10

1. Determine the fourth-order polynomial using Lagrange interpolation form that passes through all the points.
2. Use the polynomial obtained in part (a) to determine the interpolated value for $x = 3$.

Newton's Interpolating Polynomials

Newton's Interpolating Polynomials

- Newton's interpolating polynomials are a popular means of exactly fitting a given set of data points.
- The general form of an $(n-1)^{th}$ order Newton's polynomial that passes through n points is:

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + \ldots + a_n(x - x_1)(x - x_2)\ldots(x - x_{n-1})$$

- The special feature of this form of the polynomial is that the coefficients a_1 through a_n can be determined using a simple mathematical procedure.
- Determination of the coefficients does not require a solution of a system of n equations.
- Once the coefficients are known, the polynomial can be used for calculating an interpolated value at any x.

Newton's Interpolating Polynomials

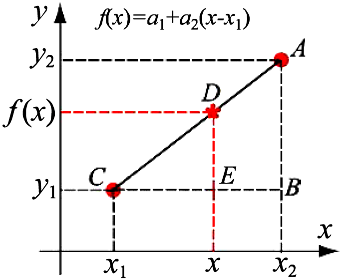
- Newton's interpolating polynomials have additional desirable features that make them a popular choice.
  - The data points do not have to be in descending or ascending order, or in any order.
  - Moreover, after the n coefficients of an $n-1$ order Newton's interpolating polynomial are determined for n given points, more points can be added to the data set and only the new additional coefficients have to be determined.

First-order Newton's polynomial

- For two given points, (x_1, y_1) and (x_2, y_2), the first-order Newton's polynomial has the form:

$$f(x) = a_1 + a_2(x - x_1)$$

It is an equation of a straight-line that passes through the points.



- The coefficients a_1 and a_2 can be calculated by considering the similar triangles

$$\frac{DE}{CE} = \frac{AB}{CB}, \quad \text{or} \quad \frac{f(x) - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3}$$

$$f(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{4}$$

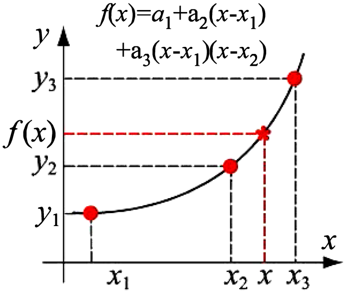
$$a_1 = y_1, \quad \text{and} \quad a_2 = \frac{y_2 - y_1}{x_2 - x_1} \tag{5}$$

Second-order Newton's polynomial

- For three given points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), the second-order Newton's polynomial has the form:

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

It is an equation of a parabola that passes through the three points.



- The coefficients a_1, a_2, and a_3 can be determined by substituting the three points in above equation.
- Substituting $x = x_1$ and $f(x_1) = y_1$ gives: $a_1 = y_1$
- Substituting the second point, $x = x_2$ and $f(x_2) = y_2$, (and $a_1 = y_1$) in above equation gives:

$$y_2 = y_1 + a_2(x_2 - x_1) \quad \text{or} \quad a_2 = \frac{y_2 - y_1}{x_2 - x_1} \qquad (6)$$

Second-order Newton's polynomial

- Substituting the third point, $x = x_3$ and $f(x_3) = y_3$ (as well as $a_1 = y_1$ and $a_2 = \dfrac{y_2 - y_1}{x_2 - x_1}$) in $f(x)$ that gives:

$$y_3 = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2) \tag{7}$$

Above equation can be solved for a_3 and rearranged to give (after some algebra):

$$a_3 = \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{(x_3 - x_1)} \tag{8}$$

- The coefficients a_1, and a_2 are the same in the first-order and second-order polynomials. This means that if two points are given and a first-order Newton's polynomial is fit to pass through those points, and then a third point is added, the polynomial can be changed to be of second-order and pass through the three points by only determining the value of one additional coefficient.

Third-order Newton's polynomial

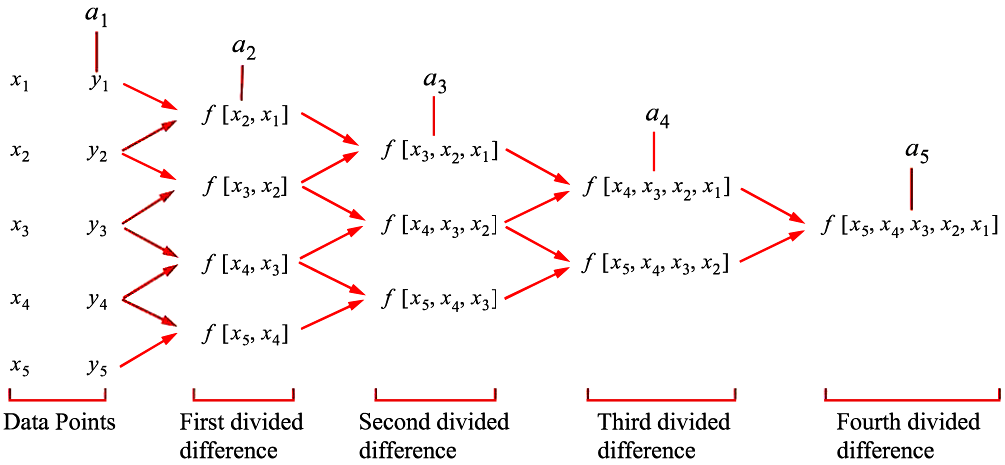
- For four given points, (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), the third-order Newton's polynomial that passes through the four points has the form:

$$f(x) = y = a_1 + a_2 (x - x_1) + a_3 (x - x_1)(x - x_2) + a_4 (x - x_1)(x - x_2)(x - x_3) \quad (9)$$

- The formulas for the coefficients a_1, a_2, and a_3 are the same as for the second order polynomial. The formula for the coefficient a_4 can be obtained by substituting (x_4, y_4), in Eq and solving for a_4, which gives:

$$a_4 = \frac{\dfrac{\left(\dfrac{y_4 - y_3}{x_4 - x_3} - \dfrac{y_3 - y_2}{x_3 - x_2}\right)}{(x_4 - x_2)} - \dfrac{\left(\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}\right)}{(x_3 - x_1)}}{(x_4 - x_1)} \quad (10)$$

A general form of Newton's polynomial and its coefficients



A general form of Newton's polynomial and its coefficients

- There is common pattern in all equations that can be clarified by defining so called divided differences.
- For two points, (x_1, y_1), and (x_2, y_2), the first divided difference, written as $f[x_2, x_1]$, is defined as the slope of the line connecting the two points:

$$f[x_2, x_1] = \frac{y_2 - y_1}{x_2 - x_1} = a_2 \tag{11}$$

- The first divided difference is equal to the coefficient a_2.

A general form of Newton's polynomial and its coefficients

- For three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) the second divided difference, written as $f[x_3, x_2, x_1]$, is defined as the difference between the first divided differences of points (x_3, y_3), and (x_2, y_2), and points (x_2, y_2), and (x_1, y_1) divided by $(x_3 - x_1)$:

$$f\left[x_3, x_2, x_1\right] = \frac{f\left[x_3, x_2\right] - f\left[x_2, x_1\right]}{x_3 - x_1} = \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{(x_3 - x_1)} = a_3 \qquad (12)$$

- The second divided difference is thus equal to the coefficient a_3.

A general form of Newton's polynomial and its coefficients

- For four points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) the third divided difference, written as $f[x_4, x_3, x_2, x_1]$, is defined as the difference between the second divided differences of points (x_2, y_2), (x_3, y_3) and (x_4, y_4), and points (x_1, y_1), (x_2, y_2), and (x_3, y_3) divided by $(x_4 - x_1)$:

$$
\begin{aligned}
f[x_4, x_3, x_2, x_1] &= \frac{f[x_4, x_3, x_2] - f[x_3, x_2, x_1]}{x_4 - x_1} \\
&= \frac{\dfrac{f[x_4, x_3] - f[x_3, x_2]}{x_4 - x_2} - \dfrac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1}}{(x_4 - x_1)} \\
&= \frac{\dfrac{\dfrac{y_4 - y_3}{x_4 - x_3} - \dfrac{y_3 - y_2}{x_3 - x_2}}{x_4 - x_2} - \dfrac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}}{(x_4 - x_1)} = a_4
\end{aligned}
$$

- The third divided difference is thus equal to the coefficient a_4.
- If more data points are given, the procedure for calculating higher differences continues in the same manner.

A general form of Newton's polynomial and its coefficients

- In general, when n data points are given, the procedure starts by calculating $(n-1)$ first divided differences.
- Then, $(n-2)$ second divided differences are calculated from the first divided differences.
- This is followed by calculating $(n-3)$ third divided differences from the second divided differences. The process ends when one nth divided difference is calculated from two $(n-1)$ divided differences to give the coefficient a_n.
- In general terms, for n given data points, (x_1, y_1), (x_2, y_2), ... , (x_n, y_n) , the first divided differences between two points (x_i, y_i), and (x_j, y_j) are given by:

$$f[x_j, x_i] = \frac{y_j - y_i}{x_j - x_i} \tag{13}$$

Notes about Newton's polynomials

- The spacing between the data points do not have to be the same.
- For a given set of n points, once the coefficients a_1 through a_n are determined, they can be used for interpolation at any point between the data points.
- After the coefficients a_1 through a_n are determined (for a given set of n points), additional data points can be added (they do not have to be in order), and only the additional coefficients have to be determined.

Examples

Example: The set of the following five data points is given:

x	1	2	4	5	7
y	52	5	-5	-40	10

1. Determine the fourth-order polynomial in Newton's form that passes through the points. Calculate the coefficients by using a divided difference table.
2. Use the polynomial obtained in part (a) to determine the interpolated value for $x = 3$.

References

- Numerical Methods Using MATLAB by Matthews and Fink, Pearson
- Applied Numerical Methods with MATLAB for Engineers and Scientists, Third Edition Steven C. Chapra, McGraw-Hill
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Thank you!